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On differential subordinations in the complex plane

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Abstract
The purpose of this work is to present a new approach to solve some problems in differential subordination theory. We also discuss the new results closely related to the generalized Briot-Bouquet differential subordination.

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1 Introduction

Let $\mathcal{H}$ denote the class of all analytic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is, if and only if the linear segment joining any two points of E lies entirely in E. A univalent function f maps $\mathbb{D}$ onto a convex domain E if and only if [1]

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad \text{for all } z \in \mathbb{D}, \tag{1.1}$$

and then f is said to be convex in $\mathbb{D}$ (or briefly convex). Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. We denote by $\mathcal{K}$ the set of all functions $f \in \mathcal{A}$ that are convex univalent in $\mathbb{D}$. We say that $f \in \mathcal{A}$ is convex of order α, $0 \le \alpha < 1$, when

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad \text{for all } z \in \mathbb{D}. \tag{1.2}$$

Functions that are convex of order α were introduced by Robertson in [2]. For two analytic functions f, g, we say that f is subordinate to g, written as $f \prec g$, if and only if there exists an analytic function ω with property $|\omega(z)| \le |z|$ in $\mathbb{D}$ such that $f(z) = g(\omega(z))$. In particular, if g is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}). \tag{1.3}$$

The idea of subordination was used for defining many of classes of functions studied in geometric function theory. For obtaining the main result, we shall use the methods of differential subordinations. The main results in the theory of differential subordinations were introduced by Miller and Mocanu in [3] and [4]. A function p, analytic in $\mathbb{D}$, is said to satisfy a first-order differential subordination if

$$\phi\bigl(p(z), zp'(z)\bigr) \prec h(z), \tag{1.4}$$

where $(p(z), zp'(z)) \in D \subset \mathbb{C}^2$ for $z \in \mathbb{D}$, $\phi : \mathbb{C}^2 \to \mathbb{C}$ and $\phi(p(z), zp'(z))$ is analytic in $\mathbb{D}$, and h is analytic and univalent in $\mathbb{D}$. The function q is said to be a *dominant* of the differential subordination (1.4) if $p \prec q$ for all p satisfying (1.4). If $\tilde{q}$ is a *dominant* of (1.4) and $\tilde{q} \prec q$ for all *dominants* q of (1.4), then we say that $\tilde{q}$ is the *best dominant* of the differential subordination (1.4).

The purpose of the present paper is to investigate interesting new results in connection with differential subordination and to improve some results obtained by Miller and Mocanu [5]. Also we remark that the reader may refer to the recent results obtained by Sokół and Nunokawa [6] as applications of differential subordination.

The following lemma will be required in our present investigation.

Lemma 1.1 ([3], [4, p.24]) *Assume that $\mathcal{Q}$ is the set of functions $f \in \mathcal{H}$ that are injective on $\overline{\mathbb{D}} \setminus E(f)$, where*

$$E(f) := \Bigl\{\zeta : \zeta \in \partial\mathbb{D} \textit{ and } \lim_{z\to\zeta} f(z) = \infty\Bigr\},$$

and are such that

$$f'(\zeta) \neq 0 \quad \bigl(\zeta \in \partial(\mathbb{D}) \setminus E(f)\bigr).$$

Let $\psi \in \mathcal{Q}$ with $\psi(0) = a$, and let

$$\varphi(z) = a + a_m z^m + \cdots$$

be analytic in $\mathbb{D}$ with

$$\varphi(z) \not\equiv a \quad \textit{and} \quad m \in \mathbb{N}.$$

If $\varphi \not\prec \psi$ in $\mathbb{D}$, then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{D} \quad \textit{and} \quad \zeta_0 \in \partial\mathbb{D} \setminus E(\psi)$$

for which

$$\varphi\bigl(|z| < r_0\bigr) \subset \psi(\mathbb{D}),$$
$$\varphi(z_0) = \psi(\zeta_0)$$

and

$$z_0\varphi'(z_0) = s\zeta_0\psi'(\zeta_0) \tag{1.5}$$

for some $s \geq m$. Moreover,

$$\mathfrak{Re}\left\{\frac{z_0\varphi''(z_0)}{\varphi'(z_0)}+1\right\} \geq s\mathfrak{Re}\left\{\frac{\zeta_0\psi''(\zeta_0)}{\psi'(\zeta_0)}+1\right\}. \tag{1.6}$$

To prove the main results, we also need the following lemma, which is a generalization of a result due to Nunokawa [7, 8].

Lemma 1.2 *Let $p(z)$ be a function analytic in $z \in \mathbb{D}$ of the form*

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n, \quad c_m \neq 0,$$

with $p(z) \neq 0$ in $|z| < 1$. If there exists a point $z_0 \in \mathbb{D}$ such that

$$\left|\arg\{p(z)\}\right| < \frac{\pi\varphi}{2} \quad \textit{for } |z| < |z_0|$$

and

$$\left|\arg\{p(z_0)\}\right| = \frac{\pi\varphi}{2}$$

for some $\varphi > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\ell\varphi,$$

where

$$\ell \geq \frac{m}{2}\left(a + \frac{1}{a}\right) \geq m \quad \textit{when } \arg\{p(z_0)\} = \frac{\pi\varphi}{2} \tag{1.7}$$

and

$$\ell \leq -\frac{m}{2}\left(a + \frac{1}{a}\right) \leq -m \quad \textit{when } \arg\{p(z_0)\} = -\frac{\pi\varphi}{2}, \tag{1.8}$$

where

$$\{p(z_0)\}^{1/\varphi} = \pm ia \quad \textit{and} \quad a > 0.$$

2 Main results

Theorem 2.1 *Let $B(z)$ and $C(z)$ be analytic in $\mathbb{D}$ with*

$$\left|\mathfrak{Im}\{C(z)\}\right| < \mathfrak{Re}\{B(z)\}. \tag{2.1}$$

If $p(z)$ is analytic in $\mathbb{D}$ with $p(0) = 1$, and if

$$\left|\arg\{B(z)zp'(z) + C(z)p(z)\}\right| < \frac{\pi}{2} + t(z), \tag{2.2}$$

where

$$t(z) = \begin{cases} \arg\{B(z)i + C(z)\} & \textit{when } \arg\{B(z)i + C(z)\} \in [0, \pi/2], \\ \arg\{B(z)i + C(z)\} - \pi/2 & \textit{when } \arg\{B(z)i + C(z)\} \in (\pi/2, \pi], \end{cases}$$

then we have

$$\mathfrak{Re}\{p(z)\} > 0, \quad z \in \mathbb{D}. \tag{2.3}$$

Proof By Lemma 1.2, if $\mathfrak{Re}\{p(z)\} > 0$ does not hold for all $z \in \mathbb{D}$, then there exists a point z_0, $|z_0| < 1$, such that

$$\left|\arg\{p(z)\}\right| < \frac{\pi}{2} \quad \text{for } |z| < |z_0|$$

and

$$\left|\arg\{p(z_0)\}\right| = \frac{\pi}{2},$$

and

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\ell,$$

where

$$\ell \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \quad \text{when } \arg\{p(z_0)\} = \frac{\pi}{2} \tag{2.4}$$

and

$$\ell \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad \text{when } \arg\{p(z_0)\} = -\frac{\pi}{2}, \tag{2.5}$$

where

$$p(z_0) = \pm ia \quad \text{and} \quad a > 0.$$

For the case $p(z_0) = ia$, $a > 0$, we are going to show that

$$\left|\arg\{B(z_0)z_0p'(z_0) + C(z_0)p(z_0)\}\right| \geq \frac{\pi}{2} + t(z_0). \tag{2.6}$$

We have

$$\begin{aligned} &\left|\arg\{B(z_0)z_0p'(z_0) + C(z_0)p(z_0)\}\right| \\ &\quad = \left|\arg\left\{p(z_0)\left[B(z_0)\frac{z_0p'(z_0)}{p(z_0)} + C(z_0)\right]\right\}\right| \\ &\quad = \left|\arg\{p(z_0)[B(z_0)i\ell + C(z_0)]\}\right|. \end{aligned} \tag{2.7}$$

By (2.1), we have $\mathfrak{Im}\{B(z_0)i\ell + C(z_0)\} > 0$. Therefore, from (2.7) we obtain

$$\begin{aligned}
&\bigl|\arg\bigl\{B(z_0)z_0p'(z_0) + C(z_0)p(z_0)\bigr\}\bigr| \\
&\quad = \frac{\pi}{2} + \begin{cases} \arg\{B(z)il + C(z)\} & \text{when } \arg\{B(z)il + C(z)\} \in [0,\pi/2], \\ \arg\{B(z)il + C(z)\} - \pi/2 & \text{when } \arg\{B(z)il + C(z)\} \in (\pi/2,\pi] \end{cases} \\
&\quad \geq \frac{\pi}{2} + \begin{cases} \arg\{B(z)i + C(z)\} & \text{when } \arg\{B(z)i + C(z)\} \in [0,\pi/2], \\ \arg\{B(z)i + C(z)\} - \pi/2 & \text{when } \arg\{B(z)i + C(z)\} \in (\pi/2,\pi] \end{cases} \\
&\quad = \frac{\pi}{2} + t(z_0).
\end{aligned} \tag{2.8}$$

This contradicts (2.2). For the case $p(z_0) = -ia$, $a > 0$, the proof runs as in the first case. □

Remark 2.1 Theorem 2.1 improves a result obtained by Miller and Mocanu [5, p.208].

Corollary 2.2 *Let $g(z)$ be analytic in $\mathbb{D}$ with $g(0) = 1$ and $|\mathfrak{Im}\{zg'(z)/g(z)\}| < 1$. If $f(z) = z + \cdots$ is analytic in $\mathbb{D}$ and*

$$\bigl|\arg\bigl\{g(z)f'(z)\bigr\}\bigr| < \frac{\pi}{2} + \nu(z), \quad z \in \mathbb{D},$$

where

$$\nu(z) = \begin{cases} \arg\{i + 1 - zg'(z)/g(z)\} & \textit{when } \arg\{i + 1 - zg'(z)/g(z)\} \in [0,\pi/2], \\ \arg\{i + 1 - zg'(z)/g(z)\} - \pi/2 & \textit{when } \arg\{i + 1 - zg'(z)/g(z)\} \in (\pi/2,\pi], \end{cases}$$

then we have

$$\mathfrak{Re}\left\{\frac{g(z)f(z)}{z}\right\} > 0, \quad z \in \mathbb{D}. \tag{2.9}$$

Proof We put $B(z) = 1$, $C(z) = 1 - zg'(z)/g(z)$, $p(z) = g(z)f(z)/z$. Then $p(z)$ is analytic in $\mathbb{D}$, $p(0) = 1$ and

$$\bigl|\mathfrak{Im}\bigl\{C(z)\bigr\}\bigr| < \mathfrak{Re}\bigl\{B(z)\bigr\} = 1.$$

Moreover, (2.2) becomes

$$\bigl|\arg\bigl\{g(z)f'(z)\bigr\}\bigr| < \frac{\pi}{2} + \nu(z), \quad z \in \mathbb{D}.$$

Hence, applying Theorem 2.1, we obtain (2.9) immediately. □

Theorem 2.3 *Let $B(z)$ and $C(z)$ be analytic in $\mathbb{D}$ with*

$$\mathfrak{Re}\left\{\frac{C(z)}{B(z)}\right\} \geq -1, \quad z \in \mathbb{D}. \tag{2.10}$$

If $p(z)$ is analytic in $\mathbb{D}$ with $p(0) = 0$, and if

$$\bigl|B(z)zp'(z) + C(z)p(z)\bigr| < \bigl|B(z) + C(z)\bigr|, \quad z \in \mathbb{D}, \tag{2.11}$$

then we have

$$\left|p(z)\right| < 1, \quad z \in \mathbb{D}. \tag{2.12}$$

Proof By Lemma 1.1, if $p(z) \not\prec z$ in $\mathbb{D}$, then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{D} \quad \text{and} \quad \zeta_0, \quad |\zeta_0| = 1$$

for which

$$p(|z| < r_0) \subset \mathbb{D}, \qquad p(z_0) = \zeta_0$$

and

$$z_0 p'(z_0) = s\zeta_0 \tag{2.13}$$

for some $s \geq 1$. Then, by (2.10), we have

$$\begin{aligned}\left|B(z_0)z_0p'(z_0) + C(z_0)p(z_0)\right| &= \left|sB(z_0) + C(z_0)\right| \\ &= \left|B(z_0)\right|\left|s + C(z_0)/B(z_0)\right| \\ &= \left|B(z_0)\right|\left|s + \mathfrak{Re}\left\{C(z_0)/B(z_0)\right\} + i\mathfrak{Im}\left\{C(z_0)/B(z_0)\right\}\right| \\ &\geq \left|B(z_0)\right|\left|1 + \mathfrak{Re}\left\{C(z_0)/B(z_0)\right\} + i\mathfrak{Im}\left\{C(z_0)/B(z_0)\right\}\right| \\ &= \left|B(z_0) + C(z_0)\right|,\end{aligned}$$

which contradicts (2.11). Therefore, $|p(z)| < 1$ in $\mathbb{D}$. □

Theorem 2.4 *Let $B(z)$ and $C(z)$ be analytic in $\mathbb{D}$ with*

$$\left|\mathfrak{Im}\left\{\frac{C(z)}{B(z)}\right\}\right| \geq \frac{1}{|B(z)|}, \quad z \in \mathbb{D}. \tag{2.14}$$

If $p(z)$ is analytic in $\mathbb{D}$ with $p(0) = 0$, and if

$$\left|B(z)zp'(z) + C(z)p(z)\right| < \sqrt{1 + \left|B(z)\right|^2\left(\left|\frac{zp'(z)}{p(z)}\right| + \mathfrak{Re}\left\{\frac{C(z)}{B(z)}\right\}\right)^2}, \quad z \in \mathbb{D}, \tag{2.15}$$

then we have

$$\left|p(z)\right| < 1, \quad z \in \mathbb{D}. \tag{2.16}$$

Proof Applying the same method as in the proof of Theorem 2.3, we have

$$\begin{aligned}\left|B(z_0)z_0p'(z_0) + C(z_0)p(z_0)\right| &= \left|p(z_0)\right|\left|B(z_0)\frac{z_0p'(z_0)}{p(z_0)} + C(z_0)\right| \\ &= \left|sB(z_0) + C(z_0)\right|\end{aligned}$$

$$\begin{aligned}
&= \left|B(z_0)\right|\left|s + C(z_0)/B(z_0)\right| \\
&= \left|B(z_0)\right|\left|s + \mathfrak{Re}\left\{C(z_0)/B(z_0)\right\} + i\mathfrak{Im}\left\{C(z_0)/B(z_0)\right\}\right| \\
&= \left|B(z_0)\right|\sqrt{\left(s + \mathfrak{Re}\left\{\frac{C(z_0)}{B(z_0)}\right\}\right)^2 + \left(\mathfrak{Im}\left\{\frac{C(z_0)}{B(z_0)}\right\}\right)^2}.
\end{aligned}$$

By (2.14), we have

$$\begin{aligned}
\left|B(z_0)z_0p'(z_0) + C(z_0)p(z_0)\right| &\geq \left|B(z_0)\right|\sqrt{\left(s + \mathfrak{Re}\left\{\frac{C(z_0)}{B(z_0)}\right\}\right)^2 + \frac{1}{|B(z_0)|^2}} \\
&= \sqrt{1 + \left|B(z_0)\right|^2\left(s + \mathfrak{Re}\left\{C(z_0)/B(z_0)\right\}\right)^2} \\
&= \sqrt{1 + \left|B(z_0)\right|^2\left(\left|\frac{z_0p'(z_0)}{p(z_0)}\right| + \mathfrak{Re}\left\{\frac{C(z_0)}{B(z_0)}\right\}\right)^2},
\end{aligned}$$

which contradicts (2.15). Therefore, $|p(z)| < 1$ in $\mathbb{D}$. □

Remark 2.2 Theorem 2.3 and Theorem 2.4 improve a result obtained by Miller and Mocanu [5, p.206].

Theorem 2.5 *Let* $p(z)$ *be analytic in* $\mathbb{D}$ *with* $p(0) = 1$ *and*

$$\mathfrak{Re}\left\{2p(z) - zp''(z)/p'(z) - 1\right\} > \alpha, \quad z \in \mathbb{D}. \tag{2.17}$$

Then we have

$$\mathfrak{Re}\left\{p(z)\right\} > \alpha, \quad z \in \mathbb{D}, \tag{2.18}$$

where $\alpha < 1$.

Proof Putting $q(z) = (p(z) - \alpha)/(1 - \alpha)$, $q(0) = 1$, we have to prove that $\mathfrak{Re}\{q(z)\} > 0$ for $z \in \mathbb{D}$. If (2.18) does not hold, then

$$q(z) \not\prec \psi(z) = (1 + z)/(1 - z), \quad z \in \mathbb{D}.$$

Hence by Lemma 1.1 there exist points

$$z_0 = r_0e^{i\theta} \in \mathbb{D} \quad \text{and} \quad \zeta_0, \quad |\zeta_0| = 1$$

for which

$$\mathfrak{Re}\left\{q(z)\right\} > 0 \quad \text{for } |z| < r_0 \quad \text{and} \quad \mathfrak{Re}\left\{q(z_0)\right\} = \mathfrak{Re}\left\{\frac{1 + \zeta_0}{1 - \zeta_0}\right\} = 0$$

and

$$z_0q'(z_0) = s\zeta_0\psi'(\zeta_0) = \frac{2s\zeta_0}{(1 - \zeta_0)^2} \tag{2.19}$$

for some $s \geq 1$. Then, by (1.6), we have

$$\mathfrak{Re}\left\{\frac{z_0 q''(z_0)}{q'(z_0)} + 1\right\} \geq s\mathfrak{Re}\left\{\frac{\zeta_0 \psi''(\zeta_0)}{\psi'(\zeta_0)} + 1\right\} = s\mathfrak{Re}\frac{1+\zeta_0}{1-\zeta_0} = 0. \tag{2.20}$$

Therefore, we have

$$\begin{aligned}
&\mathfrak{Re}\bigl\{2p(z_0) - z_0 p''(z_0)/p'(z_0) - 1\bigr\} \\
&\quad = \mathfrak{Re}\bigl\{2\bigl((1-\alpha)q(z_0) + \alpha\bigr) - z_0 q''(z_0)/q'(z_0) - 1\bigr\} \\
&\quad \leq \mathfrak{Re}\bigl\{2\bigl((1-\alpha)q(z_0) + \alpha\bigr)\bigr\} \\
&\quad = \alpha,
\end{aligned}$$

which contradicts (2.18). This completes the proof. □

Remark 2.3 Theorem 2.5 improves a result obtained by Miller and Mocanu [5, p.207].

Competing interests
The authors declare that they have no competing interests.

Authors? contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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